H-matrices

FOR DUMMIES

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Outline

What Is a ”Unbounded” Problem ?

Figure: Long distance telecommunications

Unbounded problems

Very large spatial domain compared to caracteristic length of observed phenomenon

Examples

- Wave problems
- Heat conduction
- Fluid dynamics

The "Classical" Simulation Approach

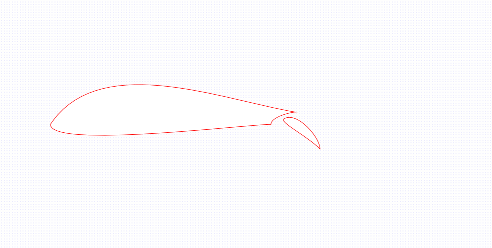

Figure: Illustration of a fluid simulation

Classic simulation workflow

- **Choose object**

The "Classical" Simulation Approach

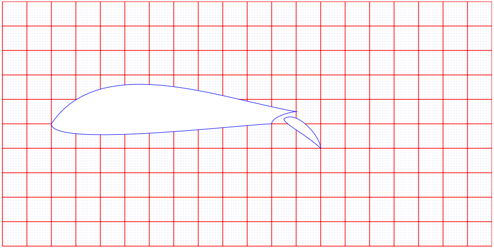

Figure: Illustration of a fluid simulation

Classic simulation workflow

- Choose object
- Discretize space

The ”Classical” Simulation Approach

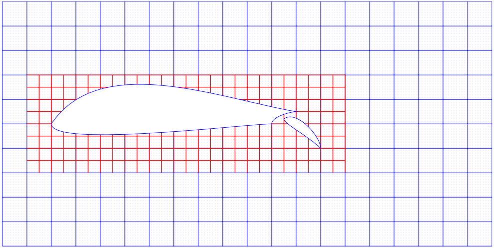

Figure: Illustration of a fluid simulation

Classic simulation workflow

- Choose object
- Discretize space
- Refine mesh

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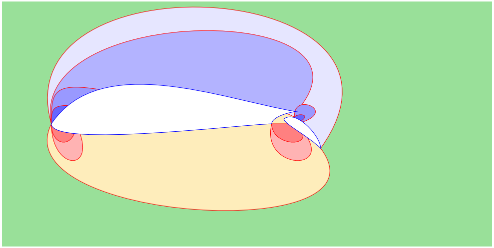

Figure: Illustration of a fluid simulation

Classic simulation workflow

- Choose object
- Discretize space
- Refine mesh
- Solve equations

How to Simulate an Unbounded Problem ?

Figure: Discretization of a large scale problem

Possible solutions

How to Simulate an Unbounded Problem ?

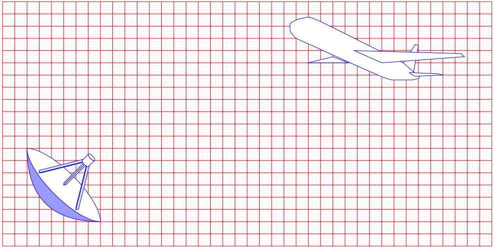

Figure: Discretization of a large scale problem

Possible solutions

- Discretize
  the whole space

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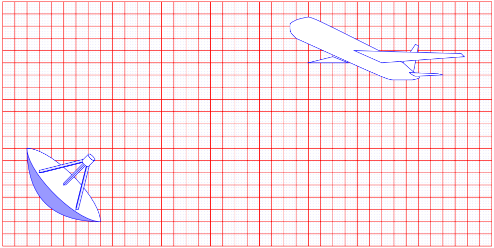

Figure: Discretization of a large scale problem

Possible solutions

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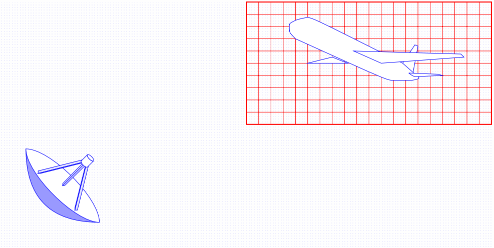

Figure: Discretization of a large scale problem

Possible solutions

- Discretize the whole space ✗
- Approximate the boundary conditions

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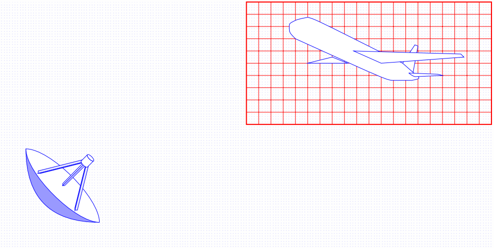

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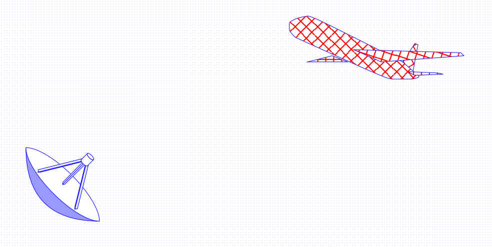

Figure: Discretization of a large scale problem

Possible solutions

- Discretize the whole space ✗
- Approximate the boundary conditions ✗
- Discretize the surface of the object

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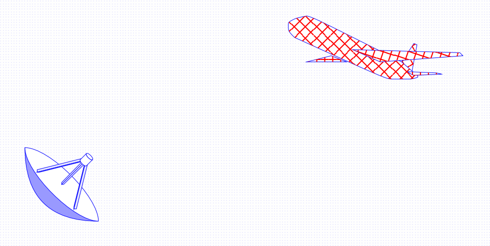

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Integral Formulation Imply Dense Equation Systems

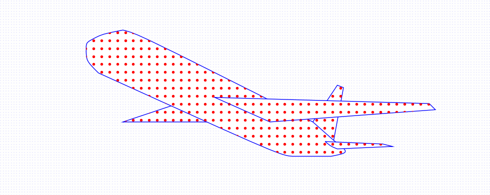


Figure: Representation of degrees of freedom on an object

Consequence of integral equations

All degrees of freedom (unknowns of problem) in contact with the same medium have an interaction

How To "Compress" a Matrix ?

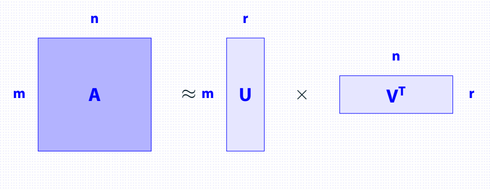


Figure: **Low-rank decomposition**

Definition (Low-rank decomposition)

$$
\begin{aligned}
\varepsilon &\in \mathbb{R} \\
\varepsilon &> 0 \\
\mathbf{A} &\in \mathcal{M}_{\mathbf{m},\mathbf{n}}(\mathbb{C}) \\
\mathbf{r} &= \operatorname{rank}(\mathbf{A}) \\
\mathbf{U} &\in \mathcal{M}_{\mathbf{m},\mathbf{r}}(\mathbb{C}) \\
\mathbf{V^T} &\in \mathcal{M}_{\mathbf{r},\mathbf{n}}(\mathbb{C}) \\
\tilde{\mathbf{A}} &= \mathbf{U} \cdot \mathbf{V^T}
\end{aligned}
$$

$$\|\mathbf{A} - \tilde{\mathbf{A}}\| < \varepsilon$$

The Hierarchical Compression Process

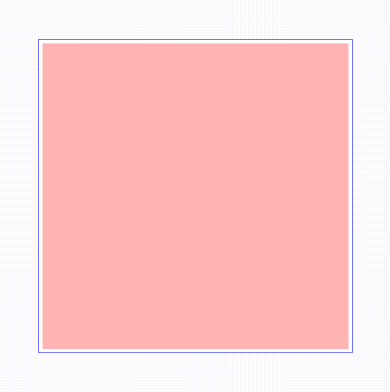
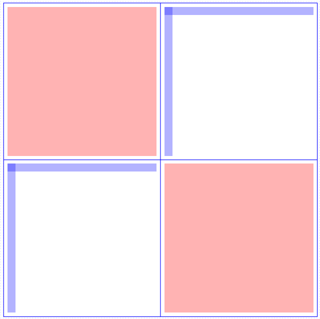


Figure: Example of $\mathcal{H}$-matrix

Process

1. **Compress**
   admissible blocks
2. **Subdivide**
   uncompressible blocks
3. **Repeat**
   until block size is too small

The Hierarchical Compression Process



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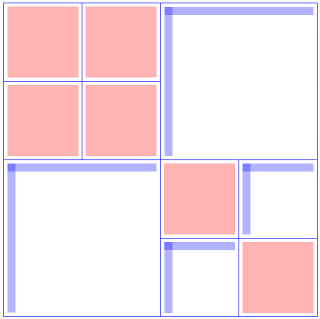

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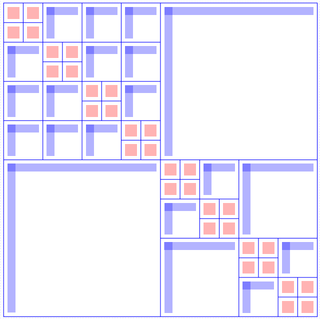

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How To Maximize Compression ?

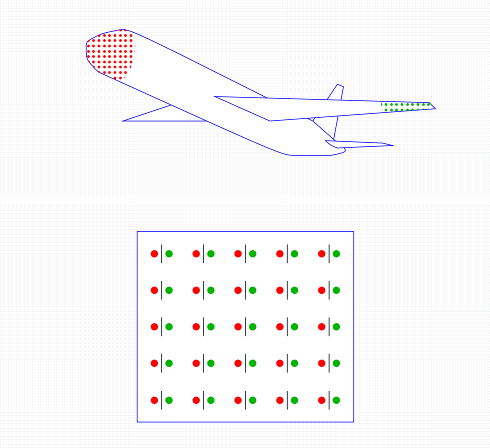
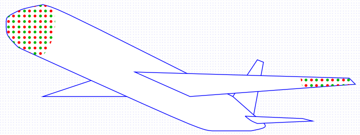


Figure: Interaction block of good quality clusters

Good clusters

- Small size
- Clearly separated
- Belong to same interface

How To Maximize Compression ?



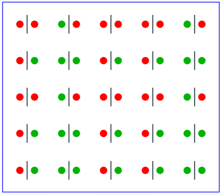


Figure: Interaction block of poor quality clusters

Good clusters

- Small size
- Clearly separated
- Belong to same interface

Bad clusters

- Large
- Close to each other
- Intertwined
- Multiple interfaces

Which Partitioning Schemes To Choose ?

Algorithm	Parallel	Balanced	Robust	Fast
Median cut	✓	✓	✗	✓
Cobblestone	✗	✓	✓	✓
K-means	✓	✗	✗	✓
Space-filling curve	✓	✓	✗	✓
Spectral partitioning	✓	✗	✓	✗
Multi-level graph partitioner	✓	✗	✓	✓

Table: Summary of partitioning schemes

Using Graphs For a Robust Partitioning

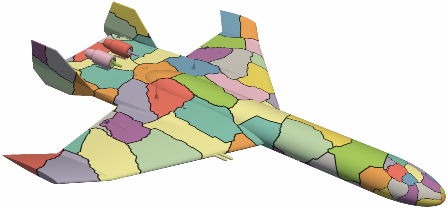
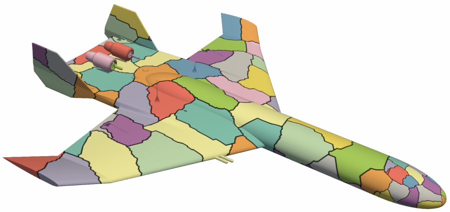


Figure: Example of graph partitioning

Advantages

- Geometry-agnostic
- Well-established partitioners
- Parallel algorithms

Using Graphs For a Robust Partitioning



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Advantages

- Geometry-agnostic
- Well-established partitioners
- Parallel algorithms

Drawbacks

- Not built for this type of tasks

Perfectly Balanced Graph Partitioning

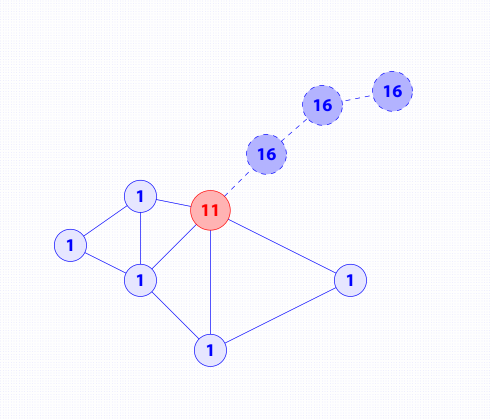

Figure: Overloaded vertex (red) and phantom partitions (dashed)

Main idea

Overload one vertex and add ”phantom partition” vertices to help the partitioner

Leads To Enhance the Quality of Partitions

Handle non-connex graphs	**Add edges in the original graph**
Fine tune position of overloaded vertex	**Geometrical or topological heuristics…**
Try other partitioning schemes	**Space filling curves, purely algebraic methods…**

Any Questions ?

Thank You For Your Attention !