In order to evaluate the electromagnetic behavior of 3D objects, we need to solve the **Maxwell equations**.

Numerical simulation of **Maxwell Equations** beforehand the conception

\[ \text{Need to solve a large and sparse linear system of equations } Ax = b \]

Requires an **efficient** and **scalable** solver

A *domain decomposition* method is used in this context, but limitations appear as the computing resources are growing:

- Increase in the number of sub-domains \(\Rightarrow\) Convergence is slower
- Increase in the size of the sub-domains \(\Rightarrow\) Increase in computational complexity

We need to investigate an alternative method to domain decomposition: **multigrid methods**.
1. Principle of multigrid methods
   • 1.1 - Principle of a multigrid cycle
   • 1.2 - Illustration of a two-level cycle on a 1D Laplace problem
   • 1.3 - Illustration of the complementarity principle
   • 1.4 - Multigrid applied to the Laplace Problem
   • 1.5 - Multigrid applied to the Helmholtz Problem

2. Smoother for Helmholtz
   • 2.1 - Introduction to the different notations
   • 2.2 - Smoother for Helmholtz
   • 2.3 - Normal equation polynomial smoother

3. Multigrid interpolation rules for Helmholtz
   • 3.1 - Introduction to the ideal framework
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   • 3.4 - Sparse approximation of the ideal interpolator

4. Benchmark

5. Conclusion and Perspectives
In this method, the computation of the solution $x$ is accelerated thanks to a hierarchy of coarse problems.

We introduce:

- $A_l$: Matrix of the level $l$
- $S_\nu^l$: $\nu$ smoother iterations on the level $l$
- $P_l$: Interpolation operator of size $n_{l-1} \times n_l$
- $R_l$: Restriction operator of size $n_l \times n_{l-1}$

In most application

$$R = P^H. \tag{1}$$
Let’s illustrate one iteration of a two-level cycle on a 1D Laplace Problem. Here, $A$ is SPD.

- Here, $x$ is chosen randomly, and $b = Ax$.
- $\tilde{x}$ approximation of $x$
- $e = x - \tilde{x}$ is the remaining information to capture
Let's illustrate one iteration of a two-level cycle on a 1D Laplace Problem. Here, $A$ is SPD.

- $\tilde{x}$ approximation of $x$, $e = x - \tilde{x}$
- The smoother captured oscillatory information
- $e$ is geometrically smooth
Let's illustrate one iteration of a two-level cycle on a 1D Laplace Problem. Here, $A$ is SPD.

- $\frac{2}{3}D^{-1}$
- $\frac{2}{3}D^{-1}$
- $P^T$ 
- $P$
- $(P^TAP)^{-1}$

- $\tilde{x}$ approximation of $x$, $e = x - \tilde{x}$
- The coarse correction captured the smooth part
- $e$ contains remaining high frequency information
Let's illustrate one iteration of a two-level cycle on a 1D Laplace Problem. Here, $A$ is SPD.

- $\tilde{x}$ approximation of $x$, $e = x - \tilde{x}$
- The smoother captured remaining oscillations
- $\tilde{x}$ is converging toward $x$!
1.3 - Illustration of the complementarity principle

Figure 2: Convergence rates of the different operators

- The **lowest** eigenvalues are damped by the **coarse correction**
- The **highest** eigenvalues are damped by the **smoother**
1.4 - Multigrid applied to the Laplace Problem

(Laplace Problem) ⇔ \[
\begin{aligned}
-\Delta u &= f \text{ on } \Omega \\
u|_{\partial \Omega} &= 0
\end{aligned}
\] (2)

Applying a 2\textsuperscript{nd} order finite difference scheme on a uniform discretization of \( \Omega \) in 1D, and using a local Fourier analysis, it yields

\[
\forall j = 1, \ldots, n , \quad \lambda_j(A) = 2h^{-2}(1 - \cos(j\pi h)) , \quad v_j(A) = [\sin(lj\pi h)]_{l=1}^n
\]

The complementarity principle is easy to satisfy

- \( \lambda_j(A) > 0 \Rightarrow \) Usual relaxation methods are efficient for capturing HF eigenvectors
- The NKS is geometrically smooth \( \Rightarrow \) Interpolation rules are easy to build in this case
1.5 - Multigrid applied to the Helmholtz problem

(Helmholtz Problem) \( \Leftrightarrow \) \[
\begin{cases}
-\Delta u - k^2 u = f & \text{sur } \Omega = [0, 1] \\
u|_{\partial \Omega} = 0
\end{cases}
\] (3)

Applying a 2\textsuperscript{nd} order finite difference scheme on a uniform discretization of \( \Omega \) in 1D, and using a local Fourier analysis, it yields

\[\forall j = 1, \ldots, n, \; \lambda_j(A) = 2h^{-2}(1 - \cos(j\pi h)) - k^2, \; \nu_j(A) = [\sin(j\pi h)]_{i=1}^n\]

\[\begin{array}{c@{\quad}c}
\lambda_i & \nu_i \\
\hline
0 & 0.1 \\
2 & 0 \\
4 & -0.1 \\
\end{array}\]

\[\begin{array}{c@{\quad}c}
\lambda_i & \nu_i \\
0 & 0.1 \\
2 & 0 \\
4 & -0.1 \\
\end{array}\]

**Figure 5:** Eigenvalues

**Figure 6:** Few of the lowest eigenvectors

**Eigenvalues are shifted**, the complementarity principle is now difficult to satisfy

- \( \lambda_j(A) \rightarrow 0 \Rightarrow \) Requires adapted smoothers: Krylov iterations, normal equations
- The NKS is now oscillatory \( \Rightarrow \) Requires new interpolation rules
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4. Benchmark

5. Conclusion and Perspectives
2.1 - Introduction to the different notations

We seek a multilevel method able to solve a 2D Helmholtz Problem defined by

\[(\text{Helmholtz Problem}) \Leftrightarrow \begin{cases} -\Delta u - k^2 u = f & \text{on } \Omega \\ \partial_n u - iku = 0 & \text{on } \partial\Omega \end{cases} \] (4)

Let the error propagation matrix for the coarse correction of a two-level cycle be

\[E_C = I - P(P^H AP)^{-1} P^H A.\] (5)

Let the error propagation matrix for the smoother be

\[E_S = I - S^{-1} A\] (6)

where \(S^{-1}\) is an approximation of \(A^{-1}\).

Furthermore, we define :

- \(V_0\) : Eigenvectors associated to the lowest eigenvalues in absolute values
- \(V_+\) : Eigenvectors associated to the highest eigenvalues in absolute values
Target: Find a smoother able to damp $V_+$, without touching $V_0$.

Problem: $V_+$ eigenvectors are either associated with negative or positive eigenvalues.

Alternative to usual multigrid smoothers:

- *Krylov methods* are good smoothers in the *indefinite* case but:
  - They minimize $||r||_2$ regardless of the eigenvalues
  - They are non-linear because of their right-hand side dependence

- *Chebyshev Polynomial Smoother* built on *normal equations* will be considered
  - Normal equations are helpful to damp both negative and positive eigenvalues
  - The Chebyshev framework is practical to find a minimum polynomial within an interval
  - This smoother has the following error propagation formula

\[
q(A^2) := I - p(A^2)A^2,
\]

(7)

giving for $Av_0 = \lambda_0 v_0 \approx 0$

\[
q(A^2)v_0 = (1 - p(\lambda_0^2)\lambda_0^2)v_0 \approx v_0
\]

(8)

→ We seek a polynomial smoother $p$ such that $q$ is minimum in a given interval.
2.3 - Normal Equation Polynomial Smoother

1. Choose an appropriate interval \( I = [x_{\text{min}}, x_{\text{max}}] \) where \( q \) must be minimum

2. Compute First Kind Chebyshev roots as best interpolation points within \( I \)

\[
    x_i := \frac{x_{\text{max}} + x_{\text{min}}}{2} + \frac{x_{\text{max}} - x_{\text{min}}}{2} \cos \left( \frac{(2i - 1)\pi}{2d} \right) \tag{9}
\]

3. Construct the polynomial using the Lagrangian formula

\[
    q(x_i) = 0 \iff p(x_i) = \frac{1}{x_i}, \quad p(x) = \sum_{j=0}^{d} \frac{1}{x_j} \prod_{i=0, i \neq j}^{d} \frac{x - x_i}{x_j - x_i} \tag{10}
\]

\[q(x) = 1 - p(x^2) x^2\]

\[q(x) = x_{\text{min}} \quad \text{and} \quad x_{\text{max}} \]

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**Figure 7:** Spectrum of the polynomial smoother error propagation matrix
2.3 - Normal Equation Polynomial Smoother

1. Choose an appropriate interval $\mathcal{I} = [x_{\text{min}}, x_{\text{max}}]$ where $q$ must be minimum

2. Compute First Kind Chebyshev roots as best interpolation points within $\mathcal{I}$

$$
    x_i := \frac{x_{\text{max}} + x_{\text{min}}}{2} + \frac{x_{\text{max}} - x_{\text{min}}}{2} \cos \left( \frac{(2i - 1)\pi}{2d} \right) \tag{11}
$$

3. Construct the polynomial using the Lagrangian formula

$$
    q(x_i) = 0 \iff p(x_i) = \frac{1}{x_i}, \quad p(x) = \sum_{j=0}^{d} \frac{1}{x_j} \prod_{i=0, i\neq j}^{d} \frac{x - x_i}{x_j - x_i} \tag{12}
$$

**Figure 8:** Spectrum of the polynomial smoother error propagation matrix
2.3 - Normal Equation Polynomial Smoother

1. Choose an appropriate interval \( I = [x_{\text{min}}, x_{\text{max}}] \) where \( q \) must be minimum.
2. Compute First Kind Chebyshev roots as best interpolation points within \( I \)
   \[
   x_i := \frac{x_{\text{max}} + x_{\text{min}}}{2} + \frac{x_{\text{max}} - x_{\text{min}}}{2} \cos \left( \frac{(2i - 1)\pi}{2d} \right) \tag{13}
   \]
3. Construct the polynomial using the Lagrangian formula
   \[
   q(x_i) = 0 \iff p(x_i) = \frac{1}{x_i}, \quad p(x) = \sum_{j=0}^{d} \frac{1}{x_j} \prod_{i=0, i \neq j}^{d} \frac{x - x_i}{x_j - x_i} \tag{14}
   \]

**Figure 9:** Spectrum of the polynomial smoother error propagation matrix.
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5. Conclusion and Perspectives
Target: Construct an interpolator $P$ containing $V_0$ in its range

From the theory, we can define, under SPD assumption of $A$, an ideal interpolator $P_*$ from coarse ($C$) and fine ($F$) variable selection operators

$$R_{(n_C \times n)} : \Omega \mapsto C \quad \text{and} \quad S^H_{(n_F \times n)} : \Omega \mapsto F,$$

such that

$$n_C + n_F = n \quad \text{and} \quad RS = 0. \quad (16)$$

This operator is defined by

$$P_* = (I - S(S^HAS)^{-1}S^HA)R^H \quad (17)$$

and minimizes the quantity

$$\mu_X = \min_P \max_{e \neq 0} \frac{\langle X(I - PR)e, (I - PR)e \rangle}{\langle Ae, e \rangle}. \quad (18)$$

where $X$ is a matrix defining the smoothing space.
Target: Construct an interpolator $P$ containing $V_0$ in its range

**Exemple 1**: Let the coarse and fine selection operators be defined by

$$R = \begin{bmatrix} 0 & I_{n_C} \end{bmatrix} \text{ and } S^T = \begin{bmatrix} I_{n_F} & 0 \end{bmatrix}.$$ \hspace{1cm} (19)

After reorganizing $A$ by coarse/fine blocks, such that

$$A = \begin{bmatrix} S^H AS & S^H AR^H \\ RAS & RAR^H \end{bmatrix} = \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix},$$

the ideal interpolator can be written

$$P^* = \begin{bmatrix} -A_{ff}^{-1} A_{fc} \\ I_{cc} \end{bmatrix}$$ \hspace{1cm} (20)

and the coarse matrix

$$A_C = P^H AP^* = A_{cc} - A_{cf} A_{ff}^{-1} A_{fc}$$ \hspace{1cm} (21)

Here we can prove

$$E^* E_F = (I - P^*(P^H AP^*)^{-1} P^H A)(I - S(S^H AS)^{-1} S^H A) = 0$$ \hspace{1cm} (22)
3.1 - Introduction to the ideal framework

Target: Construct an interpolator $P$ containing $V_0$ in its range

Example 2: Let the coarse and fine selection operators be defined by

$$R^H = V_0 \text{ and } S = V_+,$$

with $V_0/V_+$ eigenvectors associated with lowest/highest eigenvalues. It yields

$$S^H A R^H = V_+^H \begin{bmatrix} V_0 & V_+ \end{bmatrix} \begin{bmatrix} \text{Diag}(\lambda_0) & 0 \\ 0 & \text{Diag}(\lambda_+) \end{bmatrix} \begin{bmatrix} V_0^H \\ V_+^H \end{bmatrix} V_0 = 0,$$

so the ideal interpolator is defined by

$$P_* = (I - S(S^H A S)^{-1} S^H A) R^H = R^H = V_0$$

(24)

and gives the coarse matrix

$$A_C = \text{Diag}(\lambda_0)$$

(25)

Here again

$$E_* E_F = (I - P_*(P_*^H A P_*)^{-1} P_*^H A)(I - S(S^H A S)^{-1} S^H A) = 0$$

(26)
Target: Construct an interpolator $P$ containing $V_0$ in its range

Remark: The ideal framework requires $A$ to be SPD.

However the reduction viewpoint is still valid: Removing orthogonal information that the smoother captures will correct the coarse selection operator

Question: Are coarse selection operators in both previous examples good for Helmholtz?

- $R^H = \begin{bmatrix} 0 & I_{n_c} \end{bmatrix}^T$ is not relevant for Helmholtz
- $R^H = V_0$ is not practical (too expensive to compute and we need sparsity)

We can construct a better coarse selection operator $R^H$ for Helmholtz.

Idea: Construct fine interpolation rules $R_F$ of $R^H$ with a least-squares minimization strategy from smoothed random vectors approximating the near-kernel space $V_0$
First, we generate a set $K$ of $\kappa$ smoothed random vectors that approximates $V_0$.

\[
\alpha_i = \frac{\langle v_i, w \rangle}{\lambda_i}
\]

**Figure 10**: Eigen decomposition of a random vector (left) vs. smoothed random vector (right)
3.2 - Generation of a set of test vectors

First, we generate a set $K$ of $\kappa$ smoothed random vectors that approximates $V_0$. 

\[\lambda_i \] 

\[\text{mean}(\alpha_i) \]

\[0 \quad 2 \quad 4 \quad 6 \quad 8 \]

\[-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6\]

**Figure 11:** Average eigen decomposition of the total set $K$
The Least-Squares Minimization strategy consists of constructing each \textbf{fine interpolation rule} $r_i$ by minimizing the squared difference between \textbf{fine values of the near-kernel candidate vectors} and the \textbf{interpolation from their connected coarse variables} $C_i$.

\[
\forall i \in \mathcal{F}, \ r_i = \arg\min_r \sum_{l=1}^{\kappa} w_l (K_{i,l} - r \cdot K_{\mathcal{C}_i,l})^2 := \arg\min_r \mathcal{L}_i(r) \tag{27}
\]

Finding the minimum of the convex loss function $\mathcal{L}_i$ is equivalent to solving

\[
\nabla \mathcal{L}_i(r_i) = 0. \tag{28}
\]

Equation (28) can be rewritten element-wise

\[
\frac{\partial \mathcal{L}_i(r_i)}{\partial r_{ij}} = \sum_{l=1}^{\kappa} 2w_l (K_{i,l} - r_i \cdot K_{\mathcal{C}_i,l}) K_{\mathcal{C}_ij,l} = 0 , \ \forall j \in [1, \text{card}(C_i)]. \tag{29}
\]

Finally, (29) leads to a system of linear equations to solve for each fine variable $i$

\[
r_iK_{\mathcal{C}_i}WK_{\mathcal{C}_i}^H = K_iWK_{\mathcal{C}_i}^H \tag{30}
\]
Using the ideal framework and the LSM strategy, let the coarse and fine selection operators

\[ \hat{R}^H = \begin{bmatrix} R_{\mathcal{F}} & I_{\mathcal{C}} \end{bmatrix}^T, \quad \hat{S} = \begin{bmatrix} I_{\mathcal{F}} & -R_{\mathcal{F}}^H \end{bmatrix}^T, \]

where \( \hat{R}^H \) is the least squares operator with \( R_{\mathcal{F}} \) its fine variable interpolation block. We define \( A_{\mathcal{F}} := \hat{S}^H A \hat{S} \). Following the ideal interpolator definition

\[ \hat{P} := (I - \hat{S} A_{\mathcal{F}}^{-1} \hat{S}^H A) \hat{R}^H \]

(32)

However, \( \hat{P} \) requires the inverse of \( A_{\mathcal{F}} \) ! Instead

\[ \hat{P} \approx \hat{R}^H - \hat{S} X^{-1}_K \hat{S}^H A \hat{R}^H, \]

(33)

where \( X^{-1}_K \) is the best polynomial approximating \( A_{\mathcal{F}}^{-1} \) within the Krylov subspace \( K \). In practice, \( \hat{P} \) is approximated column-wise under sparsity constraints

\[ K_{\mathcal{P}_i}^m = \left\{ Z_i b_i, \ Z_i A_{\mathcal{F}} Z_i^T Z_i b_i, \ldots, (Z_i A_{\mathcal{F}} Z_i^T)^{m-1} Z_i b_i \right\}, \]

(34)

where \( b_i := \hat{S}^H A \hat{R}^H_{i,i} \), and \( Z_i : \mathbb{C}^n \to \mathbb{C}^{\text{Card}(\mathcal{P}_i)} \) is the \( i \)th column associated sparsity constraint operator, that restricts any full vector to the non-zero pattern \( \mathcal{P}_i \).
• 5-points stencil Cartesian discretization with a.b.c ($\partial_n u -iku = 0$ on $\partial \Omega$)
• 10 points per wavelength ($h = \lambda/10 \Leftrightarrow kh = 2\pi/10 \approx 0.625$)
• Multigrid parameters:
  • Normal equations polynomial smoother of degree $d = 3$
  • Number of test vectors tends to grow by a factor of 2 between each level
  • Krylov sub-size $m = 3$ in the construction of $\hat{P}$

**Figure 12:** Number of iterations following the wavenumber $k$
Figure 13: Solution $x$ of a Laplace Problem vs. a Helmholtz Problem
5 - Conclusion and perspective

Helmholtz is difficult because:

- Negative components appear and require an adapted smoother
- The near-kernel space is oscillatory and needs adapted interpolation rules

⇒ We proposed an algebraic multigrid method that reaches those requirements up to a certain limit.

Further researches will focus on:

1. Increasing the depth of the multigrid cycle
2. Sparsity has to be improved
3. Construct a better coarse selection operator that minimizes \( \sum_{i}^\kappa ||K_{:,i} - R^H R K_{:,i}||_2 \)
4. Work on an adapted framework for Helmholtz